PROJECTIVE PAIRS OF PROFINITE GROUPS

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ABSTRACT. We generalize the notion of a projective profinite group to a projective pair of a profinite group and a closed subgroup.

We establish the connection with Pseudo Algebraically Closed (PAC) extensions of PAC fields: Let M be an algebraic extension of a PAC field K. Then M/K is PAC if and only if the corresponding pair of absolute Galois groups $(\mathrm{Gal}(M), \mathrm{Gal}(K))$ is projective. Moreover any projective pair can be realized as absolute Galois groups of a PAC extension of a PAC field.

Using this characterization we construct new examples of PAC extensions of relatively small fields, e.g. unbounded abelian extensions of the rational numbers.

1. INTRODUCTION

1.1. **Projective pairs.** In the profinite category, or more generally in the pro-$\mathcal{C}$ category for some Melnikov formation of finite groups $\mathcal{C}$ (see Section 2.1), the projectivity is determined via $\mathcal{C}$ embedding problems (abbreviated as $\mathcal{C}$-EP). Namely a pro-$\mathcal{C}$ group Λ is $\mathcal{C}$-projective if and only if every $\mathcal{C}$-EP, that is, a diagram

$$
\begin{array}{c}
\Lambda \\
\downarrow{\scriptstyle\mu} \\
G \xrightarrow{\ \alpha\ } A
\end{array}
$$

in which μ, α are surjective and $G, A \in \mathcal{C}$ (in particular G, A are finite), is weakly solvable, i.e., there exists a homomorphism $\theta \colon \Lambda \to G$ such that $\alpha\theta = \mu$.

The object that this study concerns is a $\mathcal{C}$-*projective pair* (Γ, Λ). Here $\Gamma \leq \Lambda$ are pro-$\mathcal{C}$ groups with the property that every $\mathcal{C}$ *double embedding problem* (in short $\mathcal{C}$-DEP) for the pair (Γ, Λ) is *weakly solvable*. Roughly speaking a $\mathcal{C}$-DEP is a pair of two $\mathcal{C}$-EPs, one for Γ and one for Λ, which are compatible. A weak solution of a $\mathcal{C}$-DEP is a pair of compatible weak solutions of the corresponding $\mathcal{C}$-EPs. We drop

the $\mathcal{C}$ notation, if $\mathcal{C}$ is the family of all finite groups. See Section 2.2 for precise definitions.

The notion of $\mathcal{C}$-projective pairs generalizes the notion of $\mathcal{C}$-projective groups (Proposition 2.4). Moreover we give several characterizations of $\mathcal{C}$-projective pairs including the *lifting property* (Proposition 2.10) and a non-abelian cohomology characterization (Proposition 2.12).

1.2. **Projective pairs and pseudo algebraically closed extensions of fields.** The motivation for this new notion of projective pairs lies in the theory of fields. To explain this connection we start with the classical case of projective groups. Ax-Lubotzky-v.d. Dries Theorem asserts that the class of all projective groups coincides with the class of all absolute Galois groups of a special kind of fields, namely Pseudo Algebraically Closed (PAC) fields, see [5, Corollary 23.1.3]. It is important to note that there are non-PAC fields whose absolute Galois group is projective, e.g. $\mathbb{F}_q$, $\mathbb{C}(t)$, and $\mathbb{Q}^{\mathrm{ab}}$ (the latter being the maximal abelian extension of $\mathbb{Q}$).

In [8] Jarden and Razon generalize the notion of PAC fields and define PAC extensions. (See the introduction of [1] for a short survey on PAC extensions and their applications). Basing on [1] we prove an analogous connection between projective pairs and PAC extensions of PAC fields (see Theorem 1.1 below). Note that in the case M/K is algebraic we have a characterization.

Theorem 1.1.   (a) *Let M be a PAC extension of a PAC field K. Then the pair* $(\mathrm{Gal}(M), \mathrm{Gal}(K))$ *is projective.*

(b) *Let M be an algebraic extension of a PAC field K. Then M/K is PAC if and only if the restriction map $(\mathrm{Gal}(M), \mathrm{Gal}(K))$ is projective.*

(c) *Let (Γ, Λ) be a projective pair. Then there exists a separable algebraic PAC extension M of a PAC field K such that $\Gamma \cong \mathrm{Gal}(M)$, $\Lambda \cong \mathrm{Gal}(K)$.*

Note that (1) implicitly implies that the restriction map $\mathrm{Gal}(M) \to \mathrm{Gal}(K)$ is injective even if M/K is not algebraic. This is indeed true, see [1, Theorem 4.2].

In [8] Jarden and Razon prove that if K is a countable Hilbertian field and $e \geq 1$ an integer, then for almost all $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_e) \in \mathrm{Gal}(K)^e$

$$K_s(\boldsymbol{\sigma}) = \{x \in K_s \mid \forall i \ \sigma_i(x) = x\}$$

is a PAC extension of K. Moreover they prove that if M/K is PAC and L/K is algebraic, then LM/L is PAC. From these two results all the known examples of PAC extensions are derived, cf. [4] for several explicit constructions of that kind. However much is unknown, for example, for a finitely generated infinite field K we do not know if there exists a PAC extension M/K whose absolute Galois group $\mathrm{Gal}(M)$ is not finitely generated [3, Conjecture 7].

We purpose here a new group theoretic method to construct PAC extensions.

- Start with a PAC extension M/K.
- Find a PAC extension E/M. Since M is a PAC field, to find E is the purely group theoretic problem of finding a subgroup Γ of $\mathrm{Gal}(M)$ such that $(\Gamma, \mathrm{Gal}(M))$ is projective (Theorem 1.1).
- By the transitivity of PAC extensions [1, Theorem 5] E/K is PAC.

Many constructions can be generate by this method. In here we apply it to relatively small infinite extensions of any countable Hilbertian field, such as $\mathbb{Q}$.

Theorem 1.2. *Let P be a countably generated projective group, K_0 a countable Hilbertian field, and K/K_0 an abelian extension of K_0 such that*

$$\{\mathrm{ord}(\sigma) \mid \sigma \in \mathrm{Gal}(K/K_0)\} \subseteq \mathbb{N} \cup \{\infty\}$$

is unbounded. Then there exists a PAC extension M/K such that $\mathrm{Gal}(M) \cong P$.

At the moment our method does not apply for a finitely generated infinite field K. If (Γ, Λ) is a projective pair, then Γ is a quotient of Λ (Corollary 2.17). Thus

if $\text{Gal}(M)$ is finitely generated, then $\text{Gal}(E)$ constructed by the above method will also be finitely generated.

1.3. **The structure of projective pairs.** This work also contains some structural study of $\mathcal{C}$-projective pairs. For example we prove that if (Γ, Λ) is $\mathcal{C}$-projective and $\Gamma \neq \Lambda$, then

(a) the normal core of Γ is trivial, i.e. $\bigcap_{\sigma \in \Lambda} \Gamma^\sigma = 1$,

(b) $(\Lambda : \Gamma) = \infty$, and

(c) $\Lambda = N \rtimes \Gamma$, for some normal subgroup N of Λ.

It is interesting to note that the analogs (via Galois correspondence) of some of the properties of projective pairs are already known for PAC extension. By Theorem 1.1, the results for PAC fields carry over to projective pairs. (The opposite implication works only if K is PAC.)

Nevertheless we bring here group theoretic proofs for several reasons. First the aesthetic reason – the group theoretic proofs are easier. The generality reason – going via Theorem 1.1 only applies to $\mathcal{C} = $ all finite groups. Finally the strength reason – the results about projective pairs are usually stronger than the corresponding field theoretic analogs. For example the analog of (c) for a PAC extension M/K says that if G is a *finite* quotient of $\text{Gal}(M)$ that *regularly occurs* over K, then G is a quotient of $\text{Gal}(K)$ [1, Corollary 6.1].
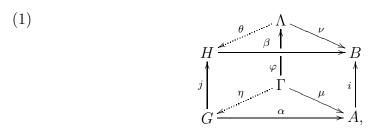
2. Definition and characterizations of projective pairs

2.1. **Melnikov formations.** Throughout this work $\mathcal{C}$ is a fixed Melnikov formation of finite groups. That means that $\mathcal{C}$ is a family of finite groups that is closed under taking fiber products and given a short exact sequence

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

we have that $A, C \in \mathcal{C}$ if and only if $B \in \mathcal{C}$. In particular, $\mathcal{C}$ is closed under direct products.

The following three families are Melnikov. The family of all finite groups; the family of all p-groups; the family of all solvable groups. More generally, if $\mathcal{S}$ is a set of simple finite groups, then the family of all finite groups whose composition factors are in $\mathcal{S}$ is a Melnikov formation.

2.2. **Double embedding problems.** Let $\Gamma \leq \Lambda$ be pro-C groups. A $\mathcal{C}$ **double embedding problem**, or in short $\mathcal{C}$**-DEP**, for the pair (Γ, Λ) is a commutating diagram

(1)

where $G, H, A, B \in \mathcal{C}$, $A \leq B$, $G \leq H$, i, j, φ are the inclusion maps, and α, μ, β, ν are surjective. Therefore a $\mathcal{C}$-DEP consists of two compatible $\mathcal{C}$-EPs: the **lower** embedding problem (μ, α) for Γ and the **higher** embedding problem (ν, β) for Λ.

In case $\mathcal{C}$ is the family of all finite groups, we omit the $\mathcal{C}$ notation and simply say that (1) is a DEP. Sometimes we abbreviate (1) and write $((\mu, \alpha), (\nu, \beta))$.

A $\mathcal{C}$-DEP is said to be **split** if α and β have sections, i.e., there exist $\alpha' \colon A \to G$ and $\beta' \colon B \to H$ for which $\alpha\alpha' = \mathrm{id}_A$ and $\beta\beta' = \mathrm{id}_B$. We emphasize that no compatibility condition on α' and β' is required, i.e. we allow that $j\alpha' \neq \beta'i$.

If the groups G, H, A, B are pro-$\mathcal{C}$, then (1) is a **pro-$\mathcal{C}$-DEP**.

Given weak solution $\eta \colon \Gamma \to G$ of the lower embedding problem and weak solution $\theta \colon \Lambda \to H$ of the higher embedding problem, we say that (η, θ) is a **weak solution** of (1) if η and θ are compatible, i.e. $\eta = \theta|_\Gamma$. Note that (η, θ) is completely determined by θ: A weak solution θ of the higher embedding problem induces a weak solution of (1) if and only if $\theta(\Gamma) \leq G$.

2.3. **The definition of $\mathcal{C}$-projective pairs.**

Definition 2.1. A pair (Γ, Λ) of pro-$\mathcal{C}$ groups is called $\mathcal{C}$-**projective** if any $\mathcal{C}$-DEP is weakly solvable.

As in the case of $\mathcal{C}$-projective groups [5, Lemma 22.3.2], the solvability property extends to pro-$\mathcal{C}$-DEPs.

Proposition 2.2. *Any pro-$\mathcal{C}$-DEP for a $\mathcal{C}$ projective pair (Γ, Λ) is weakly solvable.*

Proof. In order to solve pro-$\mathcal{C}$-DEPs for (Γ, Λ) we need to solve more general pro-$\mathcal{C}$-DEPs, in which the maps of the pro-$\mathcal{C}$-DEP are not necessarily surjective.

In case of $\mathcal{C}$-DEPs, we can solve such $\mathcal{C}$-DEPs. Indeed, assume that in (1) ν, μ are not surjective. First $\ker(\alpha), \ker(\beta) \in \mathcal{C}$ since they are normal subgroups. Next $\nu(\Gamma), \mu(\Lambda) \in \mathcal{C}$ since Γ, Λ are pro-$\mathcal{C}$. Finally $\alpha^{-1}(\nu(\Gamma)), \beta^{-1}(\mu(\Lambda)) \in \mathcal{C}$ follows from the exact sequences

$$1 \longrightarrow \ker(\alpha) \longrightarrow \alpha^{-1}(\nu(\Gamma)) \xrightarrow{\alpha} \nu(\Gamma) \longrightarrow 1$$

$$1 \longrightarrow \ker(\beta) \longrightarrow \beta^{-1}(\mu(\Lambda)) \xrightarrow{\beta} \mu(\Lambda) \longrightarrow 1.$$

Replace A, B with $\nu(\Gamma), \mu(\Lambda)$ and G, H with $\alpha^{-1}(\nu(\Gamma)), \beta^{-1}(\mu(\Lambda))$. In this new $\mathcal{C}$-DEP all the maps are surjective. So by assumption there is a weak solution.

Let us move to the more general case of pro-$\mathcal{C}$-DEP: Consider a pro-$\mathcal{C}$-DEP (1) and write $K = \ker(\beta)$. We prove the assertion in two steps.

STEP A: *Finite Kernel.* Assume K is finite. Then G is open in KG since $(KG : G) \leq |K|$. Choose an open normal subgroup $U \leq H$ for which $U \cap KG \leq G$ and $K \cap U = 1$ (note that K is finite and H is Hausdorff). Then $U \cap KG = U \cap G$. By the second isomorphism theorem (in the group UG) we have
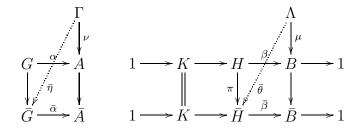
$$(KG \cap UG : G) = (U \cap (KG \cap UG) : U \cap G) = (U \cap KG : U \cap G) = 1,$$

that is to say

$$(2) \qquad\qquad (KG) \cap (UG) = G.$$

Write $\bar{H} = H/U$, let $\pi\colon H \to \bar{H}$ be the quotient map, $\bar{G} = \pi(G)$, $\bar{B} = B/\beta(U)$, $\bar{A} = A/A \cap \beta(U)$ and $\bar{\beta}\colon \bar{H} \to \bar{B}$, $\bar{\alpha}\colon \bar{G} \to \bar{A}$ the epimorphisms induced from β, α, respectively.

Since $\bar{H} \in \mathcal{C}$, there is a homomorphism $\bar{\theta}\colon \Lambda \to \bar{H}$ with $\bar{\theta}(\Gamma) \le \bar{G}$ (let $\bar{\eta} = \bar{\theta}|_\Gamma$) for which

$$
\begin{array}{ccc}
& \Gamma & \\
& \downarrow{\scriptstyle \nu} & \\
G \xrightarrow{\ \alpha\ } A & \qquad & 1 \longrightarrow K \longrightarrow H \xrightarrow{\ \beta\ } B \longrightarrow 1 \\
\downarrow{\scriptstyle \bar{\eta}}\ \ \downarrow & \qquad & \Vert \quad {\scriptstyle \pi}\downarrow\ {\scriptstyle \bar{\theta}}\ \ \downarrow{\scriptstyle \bar{\beta}} \\
\bar{G} \xrightarrow{\ \bar{\alpha}\ } \bar{A} & \qquad & 1 \longrightarrow K \longrightarrow \bar{H} \xrightarrow{\ \bar{\beta}\ } \bar{B} \longrightarrow 1
\end{array}
$$

are commutative diagrams. The right square in the right diagram is a cartesian square, since $K \cap U = 1$ ([5, Example 22.2.7(c)]). Hence we can lift $\bar{\theta}$ to $\theta\colon \Lambda \to H$ such that $\beta\theta = \mu$ ([5, Lemma 22.2.1]). We claim that $\theta(\Gamma) \le G$. Indeed,

$$
A \ge \mu(\Gamma) = \beta(\theta(\Gamma)),
$$

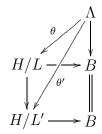hence $\theta(\Gamma) \le K\beta^{-1}(A) = K\alpha^{-1}(A) = KG$. Also,

$$
\bar{G} \ge \bar{\theta}(\Gamma) = \pi(\theta(\Gamma)),
$$

hence $\theta(\Gamma) \le UG$. Then, from (2) we have $\theta(\Gamma) \le (KG) \cap (UG) = G$, as claimed.

STEP B: *The General Case.* We use Zorn's Lemma. Consider the family of pairs (L, θ) where $L \subseteq K$ is normal in H, θ is a weak solution of the following embedding problem, and $\theta(\Gamma) \subseteq GL/L$.

$$
\begin{array}{ccc}
& \Lambda & \\
{\scriptstyle \theta}\swarrow & \downarrow & \\
1 \longrightarrow K/L \longrightarrow H/L \xrightarrow[{\scriptstyle \bar{\beta}}]{} B \longrightarrow 1.
\end{array}
$$

We say that $(L, \theta) \leq (L', \theta')$ if $L \subseteq L'$ and

$$
\begin{array}{ccc}
 & & \Lambda \\
 & \theta \swarrow & \big\downarrow \\
H/L & \dashrightarrow B & \\
\big\downarrow & \theta' \nearrow & \big\| \\
H/L' & \longrightarrow B &
\end{array}
$$

is commutative. For a chain $\{(L_i, \theta_i)\}$ we define a lower bound (L, θ) by $L = \bigcap_i L_i$ and $\theta = \varprojlim \theta_i$ (note that $\theta(\Gamma) \subset GL/L$ by [5, Lemma 1.2.2(b)]). By Zorn's Lemma there exists a minimal element (L, θ) in the family. We claim that $L = 1$. Otherwise, there is an open normal subgroup U of H with $L \not\leq U$. Part A gives (since $L/U \cap L$ is finite) a weak solution θ' of the following embedding problem such that $\theta'(\Gamma) \subseteq G(U \cap L)/(U \cap L)$.

$$
\begin{array}{ccccccc}
 & & & & & \Lambda & \\
 & & & \theta' \swarrow & & \theta \big\downarrow & \\
1 \longrightarrow & L/U \cap L & \longrightarrow & H/U \cap L & \longrightarrow & H/L. &
\end{array}
$$

Hence (L, θ) is not minimal. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.3. For an algebraic PAC extension M/K, special kind of finite double embedding problems for $(\mathrm{Gal}(M), \mathrm{Gal}(K))$ are weakly solvable [1, Proposition 4.5]. However it is unknown whether one can solve profinite double embedding problems for $(\mathrm{Gal}(M), \mathrm{Gal}(K))$. Hence Proposition 2.2 strengthens this property in case K is PAC (via Theorem 1.1).

The next result shows that $\mathcal{C}$-projective pairs generalize $\mathcal{C}$-projective groups.

Proposition 2.4. *A pro-$\mathcal{C}$ group Λ is $\mathcal{C}$-projective if and only if the pair $(1, \Lambda)$ is $\mathcal{C}$-projective.*

Proof. A $\mathcal{C}$-EP $(\mu\colon \Lambda \to A, \alpha\colon G \to A)$ for Λ is weakly solvable if and only if the $\mathcal{C}$-DEP $((1 \to 1, 1 \to 1), (\mu, \alpha))$ for $(1, \Lambda)$ is weakly solvable. This implies the first equivalence.

Let $\beta \colon H \to A$ be any epimorphism of pro-$\mathcal{C}$-group satisfying $G \leq H$ and $\beta|_G = \alpha$. Then θ is a weak solution of (μ, α) if and only if (θ, θ) is a weak solution of $((\mu, \alpha), (\mu, \beta \colon H \to A))$. $\qquad\square$

Remark 2.5. From Proposition 2.13 below it also follows that Λ is $\mathcal{C}$-projective if and only if (Λ, Λ) is $\mathcal{C}$-projective.

From technical perspective, it is important to dominate a pro-$\mathcal{C}$-DEP by a more convenient one, e.g. split pro-$\mathcal{C}$-DEP. Let us make a precise definition.

Definition 2.6. Let $\Gamma \leq \Lambda$ be pro-$\mathcal{C}$ groups and consider the following two $\mathcal{C}$-DEP for (Γ, Λ).

We say that $((\hat{\mu}, \hat{\alpha}), (\hat{\nu}, \hat{\beta}))$ **dominates** $((\mu, \alpha), (\nu, \beta))$ if there exist epimorphisms π_i, $i = 1, 2, 3, 4$ making the following diagram commutate.
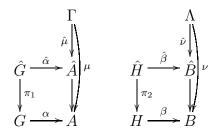

Clearly every weak solution $(\hat{\eta}, \hat{\theta})$ of the dominating $\mathcal{C}$-DEP induces a solution (η, θ) of the dominated $\mathcal{C}$-DEP by setting $\eta = \pi_1 \hat{\eta}$ and $\theta = \pi_2 \hat{\theta}$.

Lemma 2.7. *Consider a $\mathcal{C}$-DEP* (1) *for a pair* (Γ, Λ) *of pro-$\mathcal{C}$ groups. Assume that both the higher and lower embedding problems are weakly solvable. Then* (1) *is dominated by a split $\mathcal{C}$-DEP.*

Proof. Let $\theta\colon \Lambda \to H$ be a weak solution of the higher embedding problem and $\eta\colon \Gamma \to G$ a weak solution of the lower embedding problem. Choose an open normal subgroup $N \leq \Lambda$ such that $N \leq \ker(\theta)$ and $\Gamma \cap N \leq \ker(\eta)$.

Let $\hat{B} = \Lambda/N$, $\hat{A} = \Gamma/\Gamma \cap N$ and let $\hat{H} = H \times_B \hat{B}$, $\hat{G} = G \times_A \hat{A}$. Then the upper rows in the following commutative diagrams define a dominating $\mathcal{C}$-DEP.

$$
\begin{array}{ccc}
& \Gamma & \\
& \Big\downarrow{\scriptstyle\hat{\mu}} & \\
\hat{G} \xrightarrow{\hat{\alpha}} & \hat{A} & {\scriptstyle\mu} \\
\Big\downarrow{\scriptstyle\pi_1} & \Big\downarrow & \\
G \xrightarrow{\alpha} & A &
\end{array}
\qquad
\begin{array}{ccc}
& \Lambda & \\
& \Big\downarrow{\scriptstyle\hat{\nu}} & \\
\hat{H} \xrightarrow{\hat{\beta}} & \hat{B} & {\scriptstyle\nu} \\
\Big\downarrow{\scriptstyle\pi_2} & \Big\downarrow & \\
H \xrightarrow{\beta} & B &
\end{array}
$$

(Here all the maps are canonically defined.)

To finish the proof we need to show that both $\hat{\alpha}$ and $\hat{\beta}$ defined in the above diagram have sections. Let $\hat{\alpha}'\colon \hat{A} \to \hat{G}$ be defined by $\hat{\alpha}'(x) = (\eta(x), x)$, $x \in \hat{A}$, and similarly let $\hat{\beta}'\colon \hat{B} \to \hat{H}$ be defined by $\hat{\beta}'(x) = (\theta(x), x)$, $x \in \hat{B}$. Then, $\hat{\alpha}(\hat{\alpha}'(x)) = x$, $x \in \hat{A}$ and $\hat{\beta}(\hat{\beta}'(x)) = x$, $x \in \hat{B}$, i.e. both $\hat{\alpha}$ and $\hat{\beta}$ split, as needed. $\square$

Corollary 2.8. *Let* (Γ, Λ) *be a pair of pro-$\mathcal{C}$ groups and suppose that Λ is $\mathcal{C}$-projective. Then* (Γ, Λ) *is $\mathcal{C}$-projective if and only if every split $\mathcal{C}$-DEP is weakly solvable.*

Proof. Since Λ is $\mathcal{C}$-projective, Γ is also $\mathcal{C}$-projective. In other words, every finite embedding problem for Λ (resp. Γ) is weakly solvable. Lemma 2.7 implies that every $\mathcal{C}$-DEP for (Γ, Λ) is dominated by a split $\mathcal{C}$-DEP.

The converse is trivial. $\square$

Recall that for a pro-$\mathcal{C}$ group Λ to be $\mathcal{C}$-projective it is necessary and sufficient that any short exact sequence of pro-$\mathcal{C}$ groups

$$1 \longrightarrow K \longrightarrow \Delta \longrightarrow \Lambda \longrightarrow 1$$

splits. A similar characterization is given in the next result for a pair (Γ, Λ) of pro-$\mathcal{C}$ groups.

Corollary 2.9. *Let (Γ, Λ) be a pair of pro-$\mathcal{C}$ groups. Then (Γ, Λ) is $\mathcal{C}$-projective if and only if the rows of any exact commutative diagram of pro-$\mathcal{C}$ groups*

compatibly split. *That is to say, there exists a section $\beta' \colon \Lambda \to \Delta$ of β such that $\beta' \varphi(\Gamma) \leq \psi(E)$ and α' defined by $\psi \alpha' = \beta' \varphi$ is a section of α.*

Proof. Since the sections of an epimorphism $\gamma \colon M \to N$ correspond bijectively to solutions of the embedding problem $(\mathrm{id} \colon N \to N, \gamma \colon M \to N)$, the assertion follows immediately from Proposition 2.2. $\square$

2.4. **The lifting property.** The following lifting property is a key property in proving the structural results mentioned in Section 1.3. This is a stronger version of the lifting property for PAC extensions, since it applies to pro-$\mathcal{C}$-DEPs, and not only to finite DEPs.
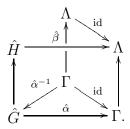
Proposition 2.10 (The lifting property). *Let* (Γ, Λ) *be* $\mathcal{C}$*-projective and consider a pro-*$\mathcal{C}$*-DEP*

(3)

for (Γ, Λ). *Then any weak solution* η *of the lower embedding problem can be lifted to a weak solution* (η, θ) *of* (3).

Proof. Let $\eta\colon \Gamma \to G$ be a weak solution of the lower embedding problem (μ, α). Define $\hat{H} = H \times_B \Lambda$ and let $\pi\colon \hat{H} \to H$ and $\hat{\beta}\colon \hat{H} \to \Lambda$ be the quotient maps. Let $\hat{G} = \{(\eta(\gamma), \gamma) \mid \gamma \in \Gamma\} \leq G \times_A \Gamma$ and $\hat{\alpha} = \hat{\beta}|_{\hat{G}}$. Then $\hat{\alpha}((\eta(\gamma), \gamma)) = \gamma$, for all $\gamma \in \Gamma$, and hence $\hat{\alpha}$ is an isomorphism. Thus $\hat{\alpha}^{-1}$ is the *unique* weak solution of the lower embedding problem of

By Proposition 2.2 there exists a weak solution $(\hat{\eta}, \hat{\theta})$ of the above pro-$\mathcal{C}$-DEP. Hence, $\hat{\eta} = \hat{\alpha}^{-1}$.

Let $\eta' = \pi|_{\hat{G}}\hat{\eta}$ and $\theta = \pi\hat{\theta}$. Then (η', θ) is a weak solution of (3). Moreover

$$\eta'(\gamma) = \pi(\hat{\alpha}^{-1}(\gamma)) = \pi((\eta(\gamma), \gamma)) = \eta(\gamma),$$

i.e. (η, θ) is a weak solution of (3), as needed. $\square$

We give two characterizations of $\mathcal{C}$-projective pairs. The first follows from the lifting property using the same argument that implied Corollary 2.9 from Proposition 2.2. The second is in terms of non-abelian cohomology.

Corollary 2.11. *Let (Γ, Λ) be a $\mathcal{C}$-projective pair and consider a diagram as in Corollary 2.9. Then any section α' of α can be lifted to a section β' of β.*

Proposition 2.12. *Let Λ be a $\mathcal{C}$-projective group and Γ a subgroup. The pair (Γ, Λ) is $\mathcal{C}$-projective if and only if for any pro-$\mathcal{C}$ group A on which Λ acts the restriction map*

$$H^1(\Lambda, A) \to H^1(\Gamma, A)$$

is surjective.

Proof. Recall that there is a natural identification between $H^1(\Lambda, A)$ and sections of the quotient map $\beta \colon A \rtimes \Lambda \to \Lambda$. More precisely, every $x \in H^1(\Lambda, A)$ induces the section β' defined by $\beta'(\lambda) = x(\lambda)\lambda$, $\lambda \in \Lambda$ and vice versa a section β' of β induces $x \in H^1(\Lambda, A)$ defined by $x(\lambda) = \beta'(\lambda)\lambda^{-1}$.

Assume (Γ, Λ) is $\mathcal{C}$-projective. Let $x \in H^1(\Gamma, A)$. It defines an embedding
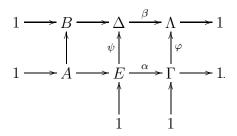
$$\psi \colon \Gamma \to A \rtimes \Gamma \le A \rtimes \Lambda, \qquad \psi(\gamma) = x(\gamma)\gamma, \ \forall \gamma \in \Gamma.$$

Corollary 2.11 applied to

$$
\begin{array}{ccccc}
A \rtimes \Lambda & \xrightarrow{\ \beta\ } & \Lambda & \longrightarrow & 1 \\
{\scriptstyle\psi}\big\uparrow & & \big\uparrow & & \\
\Gamma & \xrightarrow{\ \mathrm{id}\ } & \Gamma & \longrightarrow & 1
\end{array}
$$

(where β is the quotient map) gives $\beta' \colon \Lambda \to A \rtimes \Lambda$ such that $\beta\beta'$ is the identity on Λ and $\beta'|_\Gamma = \psi$. Then β' induces $y \in H^1(\Lambda, A)$ defined by $y(\lambda) = \beta'(\lambda)\lambda^{-1}$. Clearly the restriction of y to Γ is x.

Next assume that the restriction map $H^1(\Lambda, A) \to H^1(\Gamma, A)$ is surjective. Consider a commutative exact diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & B & \longrightarrow & \Delta & \xrightarrow{\ \beta\ } & \Lambda & \longrightarrow & 1 \\
& & \big\uparrow & & {\scriptstyle\psi}\big\uparrow & & {\scriptstyle\varphi}\big\uparrow & & \\
1 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\ \alpha\ } & \Gamma & \longrightarrow & 1. \\
& & & & \big\uparrow & & \big\uparrow & & \\
& & & & 1 & & 1 & &
\end{array}
$$

Since Λ is $\mathcal{C}$-projective, Γ is also $\mathcal{C}$-projective, and hence both rows split. Identify Δ with $B \rtimes \Lambda$ via some fixed section of β. Let α' be a section of α and $x \in H^1(\Gamma, A) \leq H^1(\Gamma, B)$ the corresponding cocycle (i.e. $x(\gamma) = \alpha'(\gamma)\gamma^{-1}$). By assumption there exists a cocyle $y \in H^1(\Lambda, B)$ satsifying $y|_\Gamma = x$. Let β' be the induced section of β. Then for all $\gamma \in \Gamma$ we have
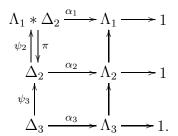
$$\beta'(\gamma) = y(\gamma)\gamma = x(\gamma)\gamma = \alpha'(\gamma).$$

Therefore $\beta'|_\Gamma = \alpha'$ and by Corollary 2.11 (Γ, Λ) is $\mathcal{C}$-projective. $\qquad\square$

Proposition 2.13 (Transitivity)**.** *Let $\Lambda_3 \leq \Lambda_2 \leq \Lambda_1$ be pro-$\mathcal{C}$ groups.*

(a) *If (Λ_3, Λ_1) is $\mathcal{C}$-projective, then (Λ_3, Λ_2) is $\mathcal{C}$-projective.*

(b) *If (Λ_3, Λ_2) and (Λ_2, Λ_1) are $\mathcal{C}$-projective, then so is (Λ_3, Λ_1).*

Proof. For (a) assume that (Λ_3, Λ_1) is $\mathcal{C}$-projective. Consider a commutative diagram

$$
\begin{array}{ccccc}
\Lambda_1 * \Delta_2 & \xrightarrow{\alpha_1} & \Lambda_1 & \longrightarrow & 1 \\
{\scriptstyle\psi_2}\big\uparrow\big\downarrow{\scriptstyle\pi} & & \big\uparrow & & \\
\Delta_2 & \xrightarrow{\alpha_2} & \Lambda_2 & \longrightarrow & 1 \\
{\scriptstyle\psi_3}\big\uparrow & & \big\uparrow & & \\
\Delta_3 & \xrightarrow{\alpha_3} & \Lambda_3 & \longrightarrow & 1.
\end{array}
$$

Here ψ_3 is injective, ψ_2 is the inclusion map, $\Lambda_1 * \Delta_2$ is the free product of Λ_1 and Δ_2 in the pro-$\mathcal{C}$ category[1], α_1 is defined by $\alpha_1|_{\Lambda_1} = \mathrm{id}$, $\alpha_1|_{\Delta_2} = \alpha_2$, and π is defined by $\pi|_{\Lambda_1} = 1$, $\pi|_{\Delta_2} = \mathrm{id}$.

There exist compatible sections β_3, β_1 of α_3, α_1 (Corollary 2.9). Let $\beta_2 = \pi\beta_1|_{\Lambda_2}$. By the above commutative diagram, β_3, β_2 are compatible sections of α_3, α_2, and thus (Λ_3, Λ_2) is $\mathcal{C}$-projective (again Corollary 2.9).

---

[1]The free product exists in the pro-$\mathcal{C}$ category. Indeed it is the maximal pro-$\mathcal{C}$ quotient of the profinite free product of Λ_1 and Λ_2

(b) easily follows from Proposition 2.12: Let A be a pro-$\mathcal{C}$ group together with a Λ_1-action. Since the restriction map $r_{1,3}\colon H^1(\Lambda_1, A) \to H^1(\Lambda_3, A)$ factors as

$$H^1(\Lambda_1, A) \xrightarrow[r_{1,2}]{\quad} H^1(\Lambda_2, A) \xrightarrow[r_{2,3}]{\quad} H^1(\Lambda_3, A)$$

and both $r_{1,2}$ and $r_{2,3}$ are surjective (Proposition 2.12) we get that $r_{1,3}$ is surjective. Consequently, (Λ_3, Λ_1) is $\mathcal{C}$-projective (again Proposition 2.12). $\qquad\square$
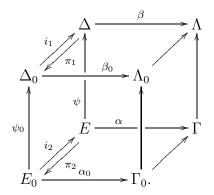
Remark 2.14. Let $\Lambda_3 \leq \Lambda_2 \leq \Lambda_1$ be pro-$\mathcal{C}$ groups. We show it does not suffice that (Λ_3, Λ_1) be $\mathcal{C}$-projective for (Λ_2, Λ_1) to be $\mathcal{C}$-projective. For this purpose look at $1 \leq p\mathbb{Z}_p \leq \mathbb{Z}_p$.

Then $(1, \mathbb{Z}_p)$ is $\mathcal{C}$-projective (Proposition 2.4) while $(p\mathbb{Z}_p, \mathbb{Z}_p)$ is not (Proposition 4.2 below).

Projective pairs behave well under taking subgroups.

Proposition 2.15. *Let (Γ, Λ) be a $\mathcal{C}$-projective pair, let $\Lambda_0 \leq \Lambda$ be a subgroup, and write $\Gamma_0 = \Lambda_0 \cap \Gamma$. Then (Γ_0, Λ_0) is $\mathcal{C}$-projective.*

Proof. Let $E_0 \leq \Delta_0$ be pro-C groups with $\psi\colon E_0 \to \Delta_0$ the inclusion map, and let $\alpha_0\colon E_0 \to \Gamma_0$ and $\beta_0\colon \Delta_0 \to \Lambda_0$ be epimorphisms satisfying $\beta_0|_{E_0} = \alpha_0$. Let $\Delta = \Lambda * \Delta_0$, $E = \Gamma_0 * E_0$ and let $i_1\colon \Delta_0 \to \Delta$ and $i_2\colon E_0 \to E$ be the corresponding injections. We define maps $\pi_1\colon \Delta \to \Delta_0$ and $\beta\colon \Delta \to \Delta_0$ by setting $\pi_1|_\Lambda = 1$, $\pi_1|_{\Delta_0} = \mathrm{id}$, $\beta|_\Gamma = \mathrm{id}$, and $\beta|_{\Delta_0} = \beta_0$. Similarly we define $\pi_2\colon E \to E_0$ and $\alpha\colon E \to \Gamma$.

By Corollary 2.9 there exist compatible sections α' and β' of α and β, respectively. Let $\alpha'_0 = \pi_2 \alpha'$ and $\beta'_0 = \pi_1 \beta'$. Then the above commutative diagram implies that α'_0 and β'_0 are compatible sections of α_0 and β_0, respectively. Hence (Γ_0, Λ_0) is $\mathcal{C}$-projective (by the same corollary). $\qquad\qquad\square$

Corollary 2.16. *Let (Γ, Λ) be a $\mathcal{C}$-projective pair and let $N \leq \Gamma$. Then there exists $M \leq \Lambda$ such that $N = \Gamma \cap M$ and $\Gamma M = \Lambda$. Moreover, if $N \triangleleft \Gamma$, then $M \triangleleft \Lambda$.*

Proof. Let $\hat{N} = \bigcap_\sigma N^\sigma$ and let $\eta \colon \Gamma \to \Gamma/\hat{N}$ be the natural quotient map. Lift η to a solution (η, θ) of the DEP

$$((\Gamma \to 1, \Gamma/\hat{N} \to 1), (\Lambda \to 1, \Gamma/\hat{N} \to 1)).$$

Let $\hat{M} = \ker(\theta)$ and $M = \theta^{-1}(N/\hat{N})$. Then (since $\eta = \theta|_\Gamma$)

$$N = \eta^{-1}(N/\hat{N}) = \Gamma \cap \theta^{-1}(N/\hat{N}) = \Gamma \cap M.$$

Since $\theta(\Gamma) = \Gamma/\hat{N}$, it follows that $\Gamma\hat{M} = \Lambda$, and in particular, $\Gamma M = \Lambda$.

To conclude the proof, note that if $N \triangleleft \Gamma$, then $N = \hat{N}$, and hence $M = \hat{M}$. So $M \triangleleft \Lambda$, as needed. $\qquad\qquad\square$

Taking $N = 1$ in the above result we get the following splitting corollary.

Corollary 2.17. *If (Γ, Λ) is $\mathcal{C}$-projective, then $\Lambda \cong M \rtimes \Gamma$ for some $M \triangleleft \Lambda$.*

3. FAMILIES OF PROJECTIVE PAIRS

3.1. **Free products.** We say that Γ is a free factor in Λ if there exists a subgroup N of Λ such that $\Lambda = \Gamma * N$.

Proposition 3.1. *Let Γ be a free factor of a $\mathcal{C}$-projective group Λ. Then (Γ, Λ) is $\mathcal{C}$-projective.*

Proof. By assumption $\Lambda = \Gamma * N$. Consider a diagram as in Corollary 2.9. Let $M = \beta^{-1}(N) \leq \Delta$ and let $\gamma = \beta|_N$. Since Λ and N are $\mathcal{C}$-projective, there

exist sections α', γ' of α, γ, respectively. Let $\beta' \colon \Lambda \to \Delta$ be the induced map. Then β' is a section of β which is compatible with α'. Thus (Γ, Λ) is $\mathcal{C}$-projective (Corollary 2.9). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For a cardinal κ let $\hat{F}_\kappa$ denote the free pro-$\mathcal{C}$ group. The following result appears in [7] (for $\kappa = \aleph_0$).

Lemma 3.2 (Haran and Lubotzky). *Let κ be an infinite cardinal and let P be a $\mathcal{C}$-projective profinite group of rank $\leq \kappa$. Then $\hat{F}_\kappa \cong P * \hat{F}_\kappa$.*

Combining the above two results yields a family of $\mathcal{C}$-projective pairs.

Corollary 3.3. *Let κ be an infinite cardinal and $\Lambda = \hat{F}_\kappa$. Then any $\mathcal{C}$-projective group Γ of rank $\leq \kappa$ can be embedded in Λ such that (Γ, Λ) is $\mathcal{C}$-projective.*

3.2. **Random finitely generated subgroups.** Let us start by fixing some notation. We write e-tuples in bold face letters, e.g., $\mathbf{b} = (b_1, \ldots, b_e)$. For a homomorphism of profinite groups $\beta \colon H \to B$, we write that $\beta(\mathbf{h}) = \mathbf{b}$ if $\beta(h_i) = b_i$ for all $i = 1, \ldots, e$. Let C be a coset of a subgroup N^e in H^e, where $N \leq \ker(\beta)$. By abuse of notation we write $\beta(C)$ for the unique element $\mathbf{b} \in B^e$ such that $\beta(\mathbf{c}) = \mathbf{b}$ for every $\mathbf{c} \in C$.

For the result of this section we need to add one condition to $\mathcal{C}$, namely we require that $\mathcal{C}$ be closed under taking subgroups. Then the pro-$\mathcal{C}$ category is closed under taking subgroups.

Proposition 3.4. *Let $\Lambda = \hat{F}_\omega$ be the free pro-$\mathcal{C}$ group of countable rank. Then for almost all $\boldsymbol{\sigma} \in \Lambda^e$ (w.r.t. the Haar measure on Λ) $(\langle \boldsymbol{\sigma} \rangle, \Lambda)$ is $\mathcal{C}$-projective.*

Proof. Let m denote the normalized Haar measure on Λ^e. Let

(4)
$$
\begin{array}{ccc}
 & & \Lambda \\
 & & \downarrow{\scriptstyle \mu} \\
H & \xrightarrow{\ \beta\ } & B
\end{array}
$$

be a $\mathcal{C}$-EP for Λ, let $\mathbf{b} \in B^e$, let $A = \langle \mathbf{b} \rangle$ be the subgroup of B generated by $\mathbf{b}$, and let $\mathbf{h} \in H^e$ be such that $\beta(\mathbf{h}) = \mathbf{b}$. Define $\Sigma = \Sigma(\mathbf{b}, \mathbf{h}, \mu, \beta) \subseteq \Lambda^e$ to be the following set.

$$\Sigma = \{\boldsymbol{\sigma} \in \Lambda^e \mid (\mu(\boldsymbol{\sigma}) = \mathbf{b}) \;\Rightarrow\; (\exists \theta \colon \Lambda \to H, \; (\beta\theta = \mu) \wedge (\theta(\boldsymbol{\sigma}) = \mathbf{h}))\},$$

that is to say, all $\boldsymbol{\sigma} \in \Lambda^e$ such that there exists a weak solution θ of (4) with $\theta(\boldsymbol{\sigma}) = \mathbf{h}$, provided $\mu(\boldsymbol{\sigma}) = \mathbf{b}$. Note that $\Sigma = (\Sigma \cap C) \cup (\Lambda^e \smallsetminus C)$, where C is the coset of $\ker(\mu)^e$ in Λ^e for which $\mu(C) = \mathbf{b}$.

We break the proof into three parts. In the first two we show that $m(\Sigma \cap C) = m(C)$, and hence $m(\Sigma) = 1$.

PART A: *Construction of solutions.* Let

$$\Delta = \{(b_i) \in H^{\mathbb{N}} \mid \beta(b_i) = \beta(b_j) \; \forall i, j \in \mathbb{N}\}.$$

It is equipped with canonical projections $\pi_i \colon \Delta \to H_i$, $i \in \mathbb{N}$. Set $\hat{\beta} \colon \Delta \to B$ by $\hat{\beta}(x) = \beta\pi_i(x)$, $x \in \Delta$. Note that $\hat{\beta}$ does not depend on i and is an epimorphism.

Let $\theta \colon \Lambda \to \Delta$ be a solution of $(\mu \colon \Lambda \to B, \hat{\beta} \colon \Delta \to B)$ (for the existence of θ see [5, Proposition 25.6.2]). Then for every $i \in \mathbb{N}$ the map $\theta_i = \pi_i\theta$ is a solution of (4). Moreover, by [2, Lemma 2.5] the set $\{\ker(\theta_i)\}$ is an independent set of subgroups of $\ker(\mu)$.

PART B: *Calculating $m(\Sigma)$.* For each $i \in \mathbb{N}$ take the coset X_i of $\ker(\theta_i)^e$ with $\theta_i(X_i) = \mathbf{h}$. Then, since

$$\mu(X_i) = \beta\theta_i(X_i) = \beta(\mathbf{h}) = \mathbf{b},$$

it follows that $X_i \subseteq C$. Moreover, Part A implies that $\{X_i \mid i \in \mathbb{N}\}$ is an independent set in C.

By the Borel-Cantelli Lemma since $\sum_i m_C(X_i) = \sum_i \frac{|B|^e}{|H|^e} = \infty$ we get that $m_C(X) = 1$. Here m_C is the normalized Haar measure on C and $X = \cap_{j=1}^{\infty} \cup_{i=j}^{\infty} X_i$. So it suffices to show that $X \subseteq \Sigma$.

Indeed, let $\boldsymbol{\sigma} \in X$. Then $\boldsymbol{\sigma} \in X_i$ for some i. It implies that θ_i is a solution of (4) and that $\theta_i(\boldsymbol{\sigma}) = \mathbf{h}$. Hence $\boldsymbol{\sigma} \in \Sigma$ and $X \subseteq \Sigma$, as desired. PART C: *Conclusion.*

Let Υ be the intersection of all $\Sigma(\mathbf{b}, \mathbf{h}, \mu, \beta)$. Since there are only countably many of them and each is of measure 1, we have $m(\Upsilon) = 1$. Let $\boldsymbol{\sigma} \in \Upsilon$ and let $\Gamma = \langle \boldsymbol{\sigma} \rangle$.

Then (Γ, Λ) is $\mathcal{C}$-projective. Indeed, consider a $\mathcal{C}$-DEP as in (1) and choose $\mathbf{h} \in G$ such that $\beta(\mathbf{h}) = \mu(\boldsymbol{\sigma})$. Then, since $\boldsymbol{\sigma} \in \Sigma(\mu(\boldsymbol{\sigma}), \mathbf{h}, \mu, \beta)$, there exists a homomorphism $\theta \colon \Lambda \to H$ such that $\theta(\Gamma) = \langle \theta(\boldsymbol{\sigma}) \rangle = \langle \mathbf{h} \rangle \leq G$. $\qquad\square$

Remark 3.5. In the above theorem we actually prove that for almost all $\boldsymbol{\sigma} \in \Lambda^e$ the pair $(\langle \boldsymbol{\sigma} \rangle, \Lambda)$ has the following stronger lifting property. For any $\mathcal{C}$-EP (1) and for any $\mathbf{h} \in G^e$ that satisfies $\alpha(\mathbf{h}) = \mu(\boldsymbol{\sigma})$ there exists a weak solution $\theta \colon \Lambda \to B$ with $\theta(\boldsymbol{\sigma}) = \mathbf{h}$.

4. RESTRICTIONS ON PROJECTIVE PAIRS

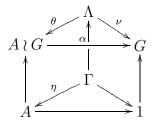**Lemma 4.1.** *Let (Γ, Λ) be a $\mathcal{C}$-projective pair and assume that $\Gamma \vartriangleleft \Lambda$. Then either $\Gamma = 1$ or $\Gamma = \Lambda$.*

Proof. Assume that both Γ and Λ/Γ are not trivial, and let $\eta \colon \Gamma \to A$ and $\nu \colon \Lambda \to G$ be epimorphism onto nontrivial $\mathcal{C}$-groups. Recall that the wreath product of A and G, denoted by $A \wr G$, is the semidirect product $A^G \rtimes G$ w.r.t. the translation action of G on A^G. The exact sequence

$$1 \longrightarrow A^G \longrightarrow A \wr G \overset{\alpha}{\longrightarrow} G \longrightarrow 1,$$

where α is the quotient map, implies that $A \wr G \in \mathcal{C}$. Identify A with the subgroup A^1 of $A \wr G$.

By the lifting property (Proposition 2.10) we can extend η to a weak solution (η, θ) of the $\mathcal{C}$-DEP

Since $\Gamma \lhd \Lambda$ we get that $A = \eta(\Gamma) = \theta(\Gamma) \lhd \theta(\Lambda)$. Let $1 \neq g \in G$, choose $\lambda \in \Lambda$ such that $\nu(\lambda) = g$, and let $h = \theta(\lambda)$. Then $h = fg$ for some $f \in A^G$. Then

$$A \leq A \cap A^h = A \cap A^{fg} = A \cap A^g = 1.$$

This contradiction implies that either $\Gamma = 1$ or $\Lambda/\Gamma = 1$, as desired. $\qquad\square$

Proposition 4.2. *Let (Γ, Λ) be a $\mathcal{C}$-projective pair. If $\Gamma \neq \Lambda$, then $\bigcap_{x \in \Lambda} \Gamma^x = 1$.*

Proof. Let $\Gamma_0 = \bigcap_{x \in \Lambda} \Gamma^x$. By Corollary 2.16 there exists Λ_0 such that $\Gamma_0 = \Lambda_0 \cap \Gamma$ and $\Gamma\Lambda_0 = \Lambda$. In particular $(\Lambda_0 : \Gamma_0) = (\Lambda : \Gamma) \neq 1$, i.e. $\Gamma_0 \neq \Lambda_0$. Moreover, by Proposition 2.15, (Γ_0, Λ_0) is a $\mathcal{C}$-projective pair. But since $\Gamma_0 \lhd \Lambda_0$ and $\Gamma_0 \neq \Lambda_0$, Lemma 4.1 implies $\Gamma_0 = 1$. $\qquad\square$

Corollary 4.3. *Let (Γ, Λ) be a $\mathcal{C}$-projective pair. Assume that Γ is open in Λ. Then $\Gamma = \Lambda$.*

Proof. Assume that $\Gamma \neq \Lambda$ (and in particular, $\Lambda \neq 1$). Since Γ is open, the normal core $\bigcap_{x \in \Lambda} \Gamma^x$ is also open. By Proposition 4.2, $\bigcap_{x \in \Lambda} \Gamma^x = 1$. Consequently $\Lambda/\bigcap_{x \in \Lambda} \Gamma^x = \Lambda$ is a nontrivial finite group. This contradicts the fact that Λ is $\mathcal{C}$-projective, and hence the assertion. $\qquad\square$

Proposition 4.4. *Let Λ be a $\mathcal{C}$-profinite group and Γ a p-Sylow subgroup. Assume that Λ has a non-abelian simple quotient that is divisible by p. Then the pair (Γ, Λ) is not $\mathcal{C}$-projective.*

Proof. Assume the contrary, i.e. (Γ, Λ) is $\mathcal{C}$-projective. Hence, by Corollary 2.17, $\Lambda = M \rtimes \Gamma$. Note that $p \nmid (\Lambda : \Gamma) = |M|$ since Γ is p-Sylow. Let $\psi \colon \Lambda \to S$ be an epimorphism onto a non-abelian simple group of order divisible by p. Then $\psi(M) \neq S$. We thus get that $\psi(M) = 1$ (since $\psi(M) \triangleleft \psi(\Lambda) = S$).

On the other hand, $\psi(\Gamma)$ is a proper subgroup of S. (Otherwise S would be a p-group, hence solvable.) The assertion now follows from the contradiction $S = \psi(\Lambda) = \psi(M)\psi(\Gamma) = \psi(\Gamma) < S$. $\square$

5. Applications to PAC extensions

In this section we shall use the following notation from [1, Section 2]. An embedding problem $(\mu \colon \mathrm{Gal}(K) \to A, \alpha \colon G \to A)$ for a field K is called **geometric** if there exists a G-extension F/E such that E is regular over K of transcendence degree 1, if we set $L = F \cap K_s$, then L is an A-extension of K, and the restriction map $\mathrm{Gal}(F/K(x)) \to \mathrm{Gal}(L/K)$ coincides with α. If in addition $E = K(x)$, then the embedding problem is called **rational**.

A weak solution of a geometric embedding problem is called **geometric** if it is induced from a K-rational place φ of E that is unramified in F.

The notion of a double embedding problem for a separable algebraic field extension M/K comes from the pair $(\mathrm{Gal}(M), \mathrm{Gal}(K))$. A double embedding problem is called **rational** if the higher embedding problem is rational. A weak solution (η, θ) of a rational double embedding problem is called **geometric** if both η and θ are geometric, say w.r.t. φ and ψ respectively, and ψ is the restriction of φ to $K(x)$ (the field defining the rational higher embedding problem). See [1, Sections 3.2 and 3.3].

5.1. **Proof of Theorem 1.1.**

Let M be a PAC extension of a PAC field K. To prove (a), we need to show that the pair $(\mathrm{Gal}(M), \mathrm{Gal}(K))$ is projective. First note that $\mathrm{Gal}(M) \cong \mathrm{Gal}(M \cap K_s)$

via the restriction map and $(M \cap K_s)/K$ is PAC ([1, Theorem 4.2]). Thus we can replace M and $M \cap K_s$, if necessary, to assume that that M/K is separable and algebraic. Let $\Gamma = \mathrm{Gal}(M)$ and $\Lambda = \mathrm{Gal}(K)$.

Since K is PAC, Λ is projective [5, Theorem 11.6.2]. By Corollary 2.8, to show that (Γ, Λ) is projective it suffices to solve a split double embedding problem (1). Over PAC fields any finite split embedding problem is rational (see e.g. [9, 6]), and hence any split DEP is rational. By [1, Proposition 4.5] there exists a weak geometric solution, and in particular a weak solution, of any finite split DEP.

For (b) assume that M is an algebraic extension of a PAC field K and that $(\mathrm{Gal}(M), \mathrm{Gal}(K))$ is projective. We have to prove that M/K is PAC.

Assume that M/K is also separable. We use [1, Proposition 4.5] which says that it suffices to geometrically solve (in the weak sense) each finite rational double embedding problem. Since $(\mathrm{Gal}(M), \mathrm{Gal}(K))$ is projective, the double embedding problem is weakly solvable. By [1, Corollary 3.4] every weak solution is geometric, and hence the assertion.

In the general case, let $N = M \cap K_s$. Then $\mathrm{Gal}(M) = \mathrm{Gal}(N)$ and N/K is separable. Then N/K is PAC. Then M/K is PAC since M/N is purely inseparable ([8, Corollary 2.3]).

For (c) let (Γ, Λ) be a projective pair. We need to construct a PAC extension M/K such that $\Gamma = \mathrm{Gal}(M)$, $\Lambda = \mathrm{Gal}(K)$.

By [5, Corollary 23.1.2], there exists a PAC field K such that $\mathrm{Gal}(K) \cong \Lambda$ (since Λ is projective). Let M be the fixed field of Γ, i.e., $\mathrm{Gal}(M) = \Gamma$. Since (Γ, Λ) is projective, by (b), M/K is PAC. $\qquad\square$

Theorem 1.1 group theoretically describes the structure of a PAC extension M of a PAC field K. Removing the condition that K is PAC gives the following more general problem.

Problem 5.1. Describe, purely group theoretically, the pairs $(\mathrm{Gal}(M), \mathrm{Gal}(K))$, where M/K is PAC (and K is arbitrary).

Note that this problem generalizes the classical problem of characterizing the class of absolute Galois groups out of all the profinite groups, and hence is much more difficult. Indeed K_s/K is PAC whenever K is infinite. Hence a description of the pair $(1, \mathrm{Gal}(K))$ clearly gives a description of $\mathrm{Gal}(K)$.

5.2. **New examples of PAC extensions.** We follow the method outlined in the introduction to construct PAC extensions.

Proposition 5.2. *Let K_0 be a field which has a PAC extension K/K_0. Assume that $\mathrm{Gal}(K)$ is free of infinite rank κ. Then for any projective group P of rank $\leq \kappa$ there exists a PAC extension M/K_0 such that $P \cong \mathrm{Gal}(M)$.*

Proof. By [1, Theorem 4.2] we can assume that K/K_0 is a separable algebraic extension. By Corollary 3.3 P embeds into $\mathrm{Gal}(K)$ in such a way that $(P, \mathrm{Gal}(K))$ is projective. Theorem 1.1 now implies that for the fixed field M of P (i.e. $\mathrm{Gal}(M) = P$), the extension M/K is PAC. Now the transitivity of PAC extensions ([1, Theorem 5]) implies that M/K_0 is PAC. $\square$

Recall that a Galois extension N/K is unbounded if the set

$$\{\mathrm{ord}(\sigma) \mid \sigma \in \mathrm{Gal}(N/K)\} \subseteq \mathbb{N} \cup \infty$$

is unbounded.

Corollary 5.3. *Let P be a projective group of at most countable rank, let K_0 be a countable Hilbertian field, and let K/K_0 be an unbounded abelian extension. Then K has a PAC extension M such that $P \cong \mathrm{Gal}(M)$.*

Proof. In the proof of [10, Proposition 3.8] it is shown that there exists a PAC extension M/K such that $\mathrm{Gal}(M) \cong \hat{F}_\omega$. Hence the previous proposition implies that there exists a PAC extension N/K with $\mathrm{Gal}(N) \cong P$. $\square$

Remark 5.4. A noteworthy special case of the last result is when K_0 is a finitely generated infinite field and K is its maximal abelian extension.

Corollary 5.5. *Let P be a countable projective group. Then there exists a Hilbertian field K and a PAC extension M/K such that $\mathrm{Gal}(M) \cong P$.*

Proof. Let $K_0 = \mathbb{Q}$ and $K = \mathbb{Q}^{\mathrm{ab}}$. Then K is Hilbertian [5, Theorem 16.11.3] and there exists a PAC extension M/K such that $\mathrm{Gal}(M) = P$ (Corollary 5.3). □

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